Math 3235 Probability Theory

1/19/23

Bayes Theorem Cont. (Sec. 1.8)

Th. (Partition Th.)

If B_i form a partition of Ω and A is an event

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)$$

A : Test positive

B_1 = Healthy $\qquad B_2$ = sick

Ω is finite

$B_i = \{i\}$ $\qquad \mathbb{P}(B_i) = \mathbb{P}(i)$

$$A \cap \{i\} = \{i\} \quad \text{if } i \in A$$
$$= \phi \quad \text{if } i \notin A$$

$$\mathbb{P}(A \mid B_i) = 1 \quad \text{if } i \in A$$
$$= 0 \quad \text{if } i \notin A$$

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i) =$$
$$= \sum_{i \in A} \mathbb{P}(i)$$

$$\mathbb{P}(A) = \mathbb{P}(A \mid B_1)\, \mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\, \mathbb{P}(B_2)$$

We want to compute $\mathbb{P}(B_2 \mid A)$ one you know

$\mathbb{P}(A \mid B_i)$ and $\mathbb{P}(B_i)$

Th. (Bayes): if B_i form a partition and A is an event

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}{\sum_j \mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)}$$

Proof:

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(B_i \cap A)}{\mathbb{P}(A)} =$$

$$= \frac{\mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}{\mathbb{P}(A)}$$

Partition Theorem

$$= \frac{\mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}{\sum_{j=1}^{n} \mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)}$$

$\blacksquare$

B_1 = Heathy = 99%

B_2 = sick = 1%

A = Test positive

A^c = The compl set of A.

$\Omega \setminus A$.

$\mathbb{P}(A \mid B_1) = 0.01$

$\mathbb{P}(A^c \mid B_1) = 0.99$

$\mathbb{P}(A \mid B_2) = 1$

$\mathbb{P}(A^c \mid B_2) = 0$

$$\mathbb{P}(B_2 \mid A) = \frac{\mathbb{P}(A \mid B_2)\,\mathbb{P}(B_2)}{\mathbb{P}(A \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\mathbb{P}(B_2)}$$

$$= \frac{1 \cdot 0.01}{0.01 \cdot 0.99 + 1 \cdot 0.01}$$

$$\approx 0.5.$$

10 000 100 sick

199 positive

1.9 Probability are Continuous.

A_i are events

$A_1 \subset A_2 \subset A_3 \subset \qquad \subset A_i \subset \ldots$

$$\mathbb{P}\left(\bigcup_{i=0}^{\infty} A_i\right) = \lim_{i \to \infty} \mathbb{P}(A_i)$$

$$\boxed{\bigcup_{i=0}^{\infty} A_i = \lim_{i \to \infty} A_i}$$

Theorem.

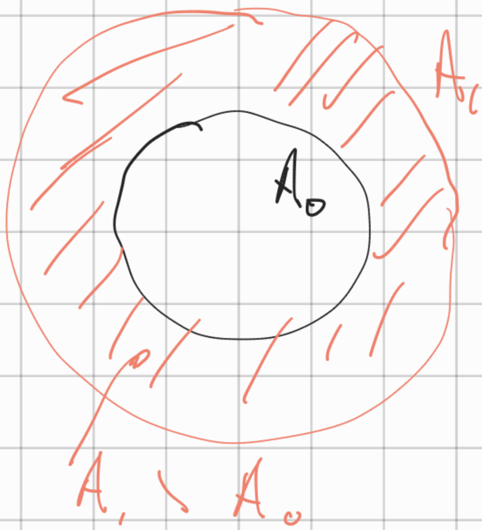
If A_i is an increasing sequence of events Then

$$\mathbb{P}\left(\bigcup_{i=0}^{\infty} A_i\right) = \lim_{i \to \infty} \mathbb{P}(A_i)$$

$$\bigcup_{i=0}^{n} A_i = A_n$$

$$A_0 \cup A_1 = A_0 \cup (A_1 \setminus A_0)$$

$$B_i = A_{i+1} \setminus A_i$$



$$\bigcup_{i=0}^{\infty} A_i = A_0 \cup \bigcup_{i=0}^{\infty} B_i \qquad B_i \cap B_j = \emptyset$$

$$\mathbb{P}\left(\bigcup_{i=0}^{\infty} A_i\right) = \mathbb{P}(A_0) + \sum_{i=0}^{\infty} \mathbb{P}(B_i)$$

$$\mathbb{P}(B_i) = \mathbb{P}(A_{i+1}) - \mathbb{P}(A_i)$$

$$\mathbb{P}\left(\bigcup_{i=0}^{\infty} A_i\right) = \mathbb{P}(A_0) + \lim_{n \to \infty} \sum_{i=0}^{n} \mathbb{P}(A_{i+1}) - \mathbb{P}(A_i)$$

$$\Big(\mathbb{P}(A_0) + \big(\mathbb{P}(A_1) - \mathbb{P}(A_0)\big)\Big) + \big(\mathbb{P}(A_2) - \mathbb{P}(A_1)$$

$$\mathbb{P}\left(\bigcup_{i=0}^{\infty} A_i\right) = \lim_{n\to\infty} \mathbb{P}(A_n) \qquad \blacksquare$$

Bayesian Statistics.

Bowl with $\underline{n}$ balls inside

Some of the balls are red and some are blue.

Prior info: K number of red balls

$p(K)$: The probability that the number of red balls is K.

$$p(K) = \frac{1}{n+1} \quad \text{for every } K.$$

Someone extracts a ball and shows it to you: it's red. Then he puts it back.

$q(k)$ probability that there k ball in the bowl known the result of the extraction.

$q(0) = 0$

$q(1) = \mathbb{P}(k \text{ red balls} \mid \text{a red one was extrac})$

$$= \frac{\mathbb{P}(\text{red} \mid k \text{ red})\,\mathbb{P}(k \text{ red})}{\sum_{i=0}^{n} \mathbb{P}(\text{red} \mid i \text{ red})\,\mathbb{P}(i \text{ red})}$$

$$= \frac{\frac{k}{n}\,\frac{1}{n+1}}{\sum_{i=0}^{n} \frac{i}{n}\,\frac{1}{n+1}}$$

$$= \frac{K}{\sum_{i=0}^{n} i} = \frac{2K}{n(n+1)}$$